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Bipartite Distance-Regular Graphs with an Eigenvalue of Multiplicity *k*

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We show that, if a bipartite distance-regular graph of valency k has an eigenvalue of multiplicity k, then it becomes 2-homogeneous. Combined with a result on bipartite 2-homogeneous distance-regular graphs by K. Nomura, we have a classification of such graphs. © 1996 Academic Press, Inc.

In the following let Γ be a connected undirected finite simple graph. For vertices x, y in Γ, let $\partial(x, y)$ denote the distance between x and y. Let d denote the maximal distance in Γ. Let $\Gamma_i(x) = \{ y \in V\Gamma \mid \partial(x, y) = i\}$ and $\Gamma(x) = \Gamma_1(x)$. A graph Γ is said to be *distance-regular* if $|\Gamma_i(x) \cap \Gamma_j(y)|$ depends only on i, j and $\partial(x, y)$. For vertices x, y with $\partial(x, y) = i$, let $c_i = |\Gamma_{i-1}(x) \cap \Gamma(y)|$, $a_i = |\Gamma_i(x) \cap \Gamma(y)|$, and let $b_i = |\Gamma_{i+1}(x) \cap \Gamma(y)|$. Remark that Γ is bipartite if and only if $a_i = 0$ $(0 \leqslant i \leqslant d)$. We call $k = |\Gamma(x)|$ $(x \in V\Gamma)$ the valency of Γ.

A bipartite distance-regular graph is said to be 2-*homogeneous* if, for every 3-tuple (x, v, w) in $V\Gamma$ with $\partial(v, w) = 2$ and $\partial(x, v) = \partial(x, w) = i$ $(2 \leqslant i \leqslant d)$, the size $|\Gamma_{i-1}(v) \cap \Gamma_{i-1}(w) \cap \Gamma(x)|$ depends only on i.

The main result of this note is the following.

THEOREM 1. *If a bipartite distance-regular graph Γ of valency k has an eigenvalue of multiplicity k, then Γ is 2-homogeneous.*

Remarks. (1) Recently, Nomura [3] shows that all bipartite 2-homogeneous distance-regular graphs are the following: (i) 2d-gon, (ii) complete bipartite graph, (iii) complement of $2 \times (k+1)$-grid, (iv) Hadamard graph, (v) antipodal 2-cover with the intersection array $\{k, k-1, k-c, c, 1; 1, c, k-c, k-1, k\}$, where $k = \gamma(\gamma^2 + 3\gamma + 1)$, $c = \gamma(\gamma + 1)$, $\gamma \geqslant 2$ (for $\gamma = 2$, 2-cover of Higman–Sims graph, and, for $\gamma \geqslant 3$, no graph is known) (vi) hypercube. Note that each of them have an eigenvalue of multiplicity k.

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(2) It is also remarked in [3] that spin models can be constructed from all such graphs.

Let Γ be a distance-regular graph of valency k, and let θ be a fixed eigenvalue of multiplicity m. We use a spherical representation (defined in [2, 1]), i.e., a mapping $^{-}: V\Gamma \to \mathbf{R}^m$ such that $(\bar{x}, \bar{y}) = u_i$ for all $x, y \in V\Gamma$ with $\partial(x, y) = i$, where $(\bar{x}, \bar{y})$ denotes the standard inner product and $\{u_0 = 1, u_1, ..., u_d\}$ is the standard sequence corresponding to θ, i.e.,

$$u_0 = 1, \quad u_1 = \theta/k, \quad \theta u_i = c_i u_{i-1} + a_i u_i + b_i u_{i+1} \quad (i = 1, ..., d-1).$$

For a subset $S \subset V\Gamma$, let $\langle \bar{S} \rangle$ denote the vectorspace spanned by $\{\bar{s} \mid s \in S\}$ and let $\hat{S} = \sum_{s \in S} \bar{s}$.

LEMMA 2. *Let Γ be a distance-regular graph. Let $z_0, z_1, ..., z_t$ be the vertices in Γ. Suppose that we have the expression of $\overline{z_0}$,*

$$\overline{z_0} = \sum_{l=1}^{t} \varepsilon_l \overline{z_l},$$

where $\varepsilon_1, ..., \varepsilon_t$ are real numbers. Then the following hold:

(1) *If, for some i, j $(1 \leqslant i, j \leqslant t)$, $\partial(z_i, z_l) = \partial(z_j, z_l)$ $(1 \leqslant i, j \leqslant t, l \neq i, j)$ and $u_{\partial(z_i, z_j)} \neq u_0$, then $\varepsilon_i = \varepsilon_j$.*

(2) *Let $z'_0, ..., z'_t$, be vertices in $V\Gamma$ such that $\partial(z_i, z_j) = \partial(z'_i, z'_j)$ $(0 \leqslant i, j \leqslant t)$. Then*

$$\overline{z'_0} = \sum_{l=1}^{t} \varepsilon_l \overline{z'_l}.$$

Proof. (1) We have

$$0 = (\overline{z_0}, \overline{z_i}) - (\overline{z_0}, \overline{z_j}) = (\varepsilon_i - \varepsilon_j)(u_0 - u_{\partial(z_i, z_j)}).$$

Hence, as $u_0 \neq u_{\partial(z_i, z_j)}$, we have the assertion.

(2) Since $(\overline{z_i}, \overline{z_j}) = (\overline{z'_i}, \overline{z'_j})$ $(0 \leqslant i, j \leqslant t)$, we have

$$0 = \left\| \overline{z_0} - \sum_{l=1}^{t} \varepsilon_l \overline{z_l} \right\|^2 = \left\| \overline{z'_0} - \sum_{l=1}^{t} \varepsilon_l \overline{z'_l} \right\|^2,$$

which is desired. ∎

In the following let Γ be a bipartite distance-regular graph of diameter d and valency k. In the case $k < 3$ and $d < 3$, Γ is clearly 2-homogeneous, so we may assume $d \geqslant 3$ and $k \geqslant 3$. Assume Γ has an eigenvalue θ of multiplicity k. Note that we have $u_0 \neq u_2$ by [1, Proposition 4.4.7] and by our assumption $d \geqslant 3$.

LEMMA 3. *For every vertex x in $V\Gamma$, $\langle \bar{\Gamma} \rangle = \langle \overline{\Gamma(x) \cup \{x\}} \rangle$.*

Proof. Since $a_1 = 0$, it follows from [4] that $\dim \langle \overline{\Gamma(x) \cup \{x\}} \rangle = k$. ∎

LEMMA 4. *There are constant α_i, β_i, γ_i $(i = 2, ..., d)$ such that, for every v, $x \in V\Gamma$ with $\partial(v, x) = i$,*

$$\bar{v} = \alpha_i \bar{x} + \gamma_i \hat{C} + \beta_i \hat{B}$$

holds, where $C = \Gamma_{i-1}(v) \cap \Gamma(x)$ and $B = \Gamma_{i+1}(v) \cap \Gamma(x)$.

Proof. As $a_i = 0$, $\Gamma(x) = C \cup B$. Hence the assertion holds by Lemma 2 and 3. ∎

Now we shall prove the theorem. Fix an integer i $(1 < i \leqslant d)$ and fix a 3-tuple (x, v, w) in $V\Gamma$ with $\partial(v, w) = 2$ and $\partial(x, v) = \partial(x, w) = i$. We partition $\Gamma(x)$ as $\Gamma(x) = A \cup B \cup C \cup D$, where

$$A = \Gamma(x) \cap \Gamma_{i-1}(v) \cap \Gamma_{i-1}(w), \qquad B = \Gamma(x) \cap \Gamma_{i-1}(v) \cap \Gamma_{i+1}(w),$$

$$C = \Gamma(x) \cap \Gamma_{i+1}(v) \cap \Gamma_{i-1}(w), \qquad D = \Gamma(x) \cap \Gamma_{i+1}(v) \cap \Gamma_{i+1}(w).$$

Clearly we have $|A| + |B| = |A| + |C| = c_i$, so that $|B| = |C|$. By Lemma 4, we can write

$$\bar{v} = \alpha_i \bar{x} + \gamma_i (\hat{A} + \hat{B}) + \beta_i (\hat{C} + \hat{D}),$$

$$\bar{w} = \alpha_i \bar{x} + \gamma_i (\hat{A} + \hat{C}) + \beta_i (\hat{B} + \hat{D}),$$

where α_i, β_i, and γ_i depend only on i. Hence

$$\|\bar{v} - \bar{w}\|^2 = (\gamma_i - \beta_i)^2 \, \|\hat{B} - \hat{C}\|^2.$$

Here we have $\|\bar{v} - \bar{w}\|^2 = 2(u_0 - u_2)$ and

$$\begin{aligned} \|\hat{B} - \hat{C}\|^2 &= \|\hat{B}\|^2 + \|\hat{C}\|^2 - 2(\hat{B}, \hat{C}) \\ &= 2(|B| \, u_0 + |B| \, (|B| - 1) u_2) - 2 \, |B|^2 \, u_2 \\ &= 2 \, |B| \, (u_0 - u_2). \end{aligned}$$

Since $u_0 \neq u_2$, we have $(\gamma_i - \beta_i)^2 \, |B| = 1$ and $|A| = c_i - |B| = c_i - (\gamma_i - \beta_i)^{-2}$. This means the size of $\Gamma_{i-1}(v) \cap \Gamma_{i-1}(w) \cap \Gamma(x)$ depends only on i, so that Γ becomes 2-homogeneous.

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